Vacuum Polarization of a Scalar Field in a Rectangular Waveguide

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Abstract

An analysis of the one-loop vacuum fluctuations associated with a massless scalar field, confined in the interior of a rectangular infinitely long waveguide is presented. To identify the infinities of the vacuum fluctuations we are using different analytic regularization procedures, instead of the usual point-splitting Green's function method. The infinities which occur in $\langle\varphi^2(x)\rangle$ fall into two distinct classes: local divergences that are renormalized by the introduction of bulk counterterms and also surface and edges divergences that demand countertems concentrated on the boundaries.

Detailed form of the surface and edge divergences are presented. Different possible solutions that can eliminate these divergences are discussed and also possible experimental test of our results.

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1 Introduction

Considerable progress has been made during the last thirty years in our understanding of the problem and consequences of assuming that quantum fields are defined in the presence of classical boundaries. In 1948 Casimir [1] predicted that uncharged, parallel, perfectly conducting plates should attract with a force per unit area, $F(L) \propto \frac{1}{L^4}$, where L is the distance between the plates. This force can be interpreted as the manifestation of the zero point energy of the electromagnetic field in the presence of the plates. Complete reviews of this effect can be found in refs. [2] [3] [4].

Although the discussions in the literature are devoted to the global and also local approach, global quantities are more acessible to experiments. It is well known that there are two quantities which might be expected to correspond to the total renormalized energy of quantum fields [5]. The first one is called the mode sum energy and its definition is

$$\langle E \rangle_{ren}^{mode} = \int_0^\infty d\omega \, \frac{1}{2}\omega [N(\omega) - N_0(\omega)], \tag{1}$$

where $\frac{1}{2}\omega$ is the zero point energy for each mode, $N(\omega)d\omega$ is the number of modes with frequencies between ω and $\omega + d\omega$ in the presence of boundaries and $N_0(\omega)d\omega$ is the corresponding quantity evaluated in empty space. The above equation gives the renormalized sum of the zero point energy for each mode. The second one is the volume integral of the renormalized energy density $\langle E \rangle_{ren}^{vol}$ obtained by the Green's functions method [6] [7].

For special configurations where the modes and the eigenfrequencies of the electromagnetic field in the presence of the boundaries can be found, it is possible to find the Casimir energy and the Casimir force. Many papers have been made to find the Casimir energy and the Casimir force for different boundaries configurations. For the case of cylinder geometry Balian and Duplantier and also others calculated the Casimir energy of the electromagnetic field [8] [9] [10] [11]. For the important case of the spherical shell, in the presence of scalar or fermionic fields Bender and Hays [12], studying the global problem, found the renormalized zero point energy of these fields assuming that the fields are confined in a spherical region of the space. Many years before Bender and Hays, also studying the global problem, Boyer [13] and Davies [14] obtained also the Casimir energy of an electromagnetic field in the presence of a perfectly conducting spherical shell. A systematic study of the spherical shell configuration was made by Milton [15]. This author calculated the zero point energy for gluons and fermions respectively, assuming that the fields are confined in the interior of the shell. Further, he studied the Casimir energy of massless fermions in the presence of a spherical shell, but in this case taking into account the external modes of the fermions [16]. More recently, working in a generic flat D-dimensional spacetime Bender and Milton [17] obtained the Casimir energy associated with a massless scalar field confined in the interior of a hypersphere. Still studying the spherical configuration, more recently Romeo [18] investigated the Casimir energy of a massless scalar field and for QED assuming that the field is confined in the interior of the spherical shell, i.e. treating only the global problem. Finally Bordag et.al. [19] have studied the Casimir energy associated with a massive scalar field in the presence of a spherical shell assuming that the interior and the exterior modes give contributions to the energy.

Although global effects are more directly accessible to experiments, it is important to compute the renormalized vacuum expectation value of the stress-energy tensor near classical boundaries. Additional motivation to find the renormalized vacuum field fluctuations $\langle\varphi^2(x)\rangle_{ren}$, the local energy density and the other components of the renormalized vacuum expectation value of the stress-energy tensor will be discussed in the conclusions. There, we will present some experiments that can be sensitive to the distortion of the vacuum field fluctuations by the presence of classical boundaries.

We can add other motivations to find the renormalized two-point function and the renormalized vacuum expectation value of the stress-energy tensor in the waveguide configuration. It is commonly accepted that, the understanding of the renormalization of the stress-energy tensor of quantum fields in presence of classical boundaries should throw light on the more difficult case, where there is the added complication of local curvature effects [20].

The aim of this paper is to generalize the results obtained by Fulling [21] and others, where the renormalized vacuum fluctuations, i.e. $\langle\varphi^2(x)\rangle_{ren}$ of a massless scalar field, is calculated between two parallel plates. We analyze the same problem introducing edges in the boundaries where the field satisfying some classical boundary condition is defined, i.e. we are interested to obtain the renormalized one-loop vacuum fluctuations associated with a massless scalar field defined in the interior of a infinitely long rectangular waveguide. With the renormalized vacuum fluctuations, to find the renormalized vacuum expectation value of the stress-energy tensor we have only to apply certain differential operator in the renormalized two-point function, and finally the coincident point limit must be taken. Note that in the case of a massless, minimally coupled scalar field the renormalized vacuum expectation value of the stress-energy tensor diverges as we approach the boundary. For a conformally coupled scalar field the renormalized vacuum expectation value of the stress-energy tensor is finite near the boundary, for the case of flat boundaries.

Although the main interest in the literature is the global Casimir energy, from which we can derive the force on the boundaries, it has often been suggested the necessity of studying the local problem since local results contain more information than the global ones. A few years ago, Actor [22] and also Actor and Bender considered this kind of problem [23]. These authors studied the use of the zeta function method to find effective action associated with a scalar field defined in the interior of the infinitely long waveguide. The local problem has been considered also in the literature by other authors. For the case of the Casimir configuration, i.e. paralel plates, Brown and Maclay [6] obtained the local form of the Casimir result, i.e. the renormalized vacuum expectation value of the stress-energy tensor associated with a electromagnetic field. Deutsch and Candelas [5] evaluated the renormalized stress-energy tensor associated with a conformally coupled scalar field and also electromagnetic field in the wedge shaped region formed by two plane boundaries. Recently, Brevik et al [24] repeated these calculations using the Schwinger's source theory. The results that they obtained agree with those of Deutsch and Candelas. In the spherical geometry, the local problem was investigated by Olaussen and Ravndal [25]. These authors studied the vacuum fluctuations of a electromagnetic field within a perfectly conducting spherical cavity. They obtained that the vacuum expectation value of the squared electric and magnetic fields i.e. $\left\langle 0|\vec{E}^2|0\right\rangle_{ren}$ and $\left\langle 0|\vec{B}^2|0\right\rangle_{ren}$, diverge as one approaches the boundary. This result has been obtained by DeWitt [26], Deutsch and Candelas [5] and also Kennedy et al [27]. The generalization of this result to the non-abelian gluon fields in the MIT bag model was achieved by Olaussen and Ravndal [28] and also Milton [29]. It has often been suggested that these surface divergences are related with the uncertainly relation between the field and the canonical conjugate momentum associated with the field [25] [30] [31].

As we stressed, we are interested to study the one-loop vacuum fluctuations associated with a scalar field developing a machinery adequate to deal with rectangular geometries. In others words, we are interested in calculating the renormalized vacuum fluctuations associated with a massless scalar field $\langle\varphi^2(x)\rangle_{ren}$, confined in the interior of a infinitely rectangular waveguide. As we will see, it will appear surface and edge divergences requiring the introduction of surface and edge counterterms. Some previous papers going in this direction have been made by Symanzik and others [30] [32] [33]. Preliminar calculations of the renormalized vacuum expectation value of the stress-energy tensor in the rectangular waveguide was performed by Dowker and Banach [34]. More recently Hacyan et al [35] and also Maclay [36] studied the vacuum fluctuations of the electromagnetic field and the Casimir force in the interior of a rectangular waveguide. The fundamental problem in rectangular cavities is the lack of translational invariance where Green's functions are expressed in terms of infinite double summations.

For interacting quantum field theory, the problem is more complicated. In flat spacetime for systems where some dimensions are compactified but translational invariance is maintained

Toms [37] and also Birrel and Ford [38] studying self-interacting scalar fields showed that all the counterterms of an interacting theory are independent of the compactified spatial size up to the two-loops approximation. Although periodic or anti-periodic boundary conditions have already been studied extensively in the literature, the quantity of papers studying interacting quantum field theory in domains where translational invariance is lost is rather small. For translational invariant systems, using Poincaré invariance one expects that overlapping divergences do not prevent the implementation of the renormalization program. In systems where Poincaré invariance does not hold, these proofs do not apply, and one must show that it is still possible to implement the renormalization program. We meet also a technical difficulty, since the presence of geometric restrictions makes Feynman diagrams harder to compute than ordinary quantum field theory in unbounded systems. For translational invariant systems, we can go from coordinate space to momentum space representation which is a more convenient framework to analyse the divergences of the system. Translational invariance is preserved for momentum conservation conditions. Since our system possesses translational invariance along two directions a more convenient representation to the Green's function is a mixed $(x, y, \vec{p})$ representation, where $\vec{p} = (p_3, p_4)$ is a two-component vector. This mixed representation was derived in many different papers and more recently was discussed by Fosco and Svaiter [39].

We assume that the scalar field is confined in the interior of the waveguide, since we meet a basic difficulty to find the exterior modes. One can avoid this difficulty assuming a "bag" configuration. Although the exterior modes problem is still unsolved, as was stressed by Actor and Bender, this problem can be brought under control introducing auxiliary configurations in the exterior region [23]. One can determine how these auxiliar configurations affect the renormalization problem. This approach was used by Svaiter and Svaiter [40] studying a massless scalar field in a $3D$ spacetime. These authors investigated the global problem using the Casimir approach, i.e adding auxiliary configurations. It is clear that these auxiliar configurations should not give spurious contributions to the finite renormalized energy and can solve the exterior modes problem of a rectangular infinitely long waveguide if we are interested in the global approach. A recent investigation of $\langle E\rangle_{ren}^{mode}$ in rectangular geometries was given also in ref. [41]. A seminal paper studying this kind of geometries was made by Ambjorn and Wolfram [42], and more recently Milton and Ng studied the Casimir effect in $(2+1)$ Maxwell-Chern-Simons electrodynamics in a rectangular domain [43]. The local boundary effects has a counterpart in the global approach. If one is interested in the global problem, the zeta function method (in the global version) can be used. In this case the divergences that appear in the regularized energy are related with Weyl's theorem, i.e. the asymptoptic distribution of eigenvalues of some elliptic differential operator [44] [45] [46]. Note that at principle it is possible to find the asymptotic distribution of the eigenvalues for an arbitrary domain M. For Neumann and Dirichlet boundary conditions the first terms are proportional to the three volume, the surface are of ∂M and the extrinsic curvature. All these informations are important only to known the polar structure of the regularized energy in the case of the global problem.

Going back to the local problem, to gain a clear undestanding of the physical meaning of the divergences that appear near classical boundaries, Fosco and Svaiter studied the one-loop renormalization of the scalar anisotropic model assuming that the scalar fields were defined in a d-dimensional Euclidean space where the first $(d-1)$ coordinates are unbounded, while the last one, that we call z lies in the interval $[0, L]$ [39]. These authors analysed the one-loop vacuum fluctuations of a massive scalar field assuming different boundary conditions on the plates, namely DD and also NN, where D denotes a Dirichlet and N a Newmann boundary condition respectively. They obtained two different results. The first one has been obtained previously by many authors, and is the fact that to renormalize the theory we have to introduce surface counterterms. The second one is the fact that the tadpole graph for DD and for NN have the same z dependent part in modulus but with opposite signs. This second result has been obtained by DeWitt [26] and also Deutsch and Candelas [5]. Fosco and Svaiter investigated the relevance of this fact to the elimination of the surface divergences.

Here we are interested in calculating local quantities in the presence of the edges. As was stressed by Dowker and Kennedy [47], in the study of the local problem it is necessary to present a closed form for the analytic continuation of the local zeta function in the rectangle. In a previous paper we found this analytic extension [48]. Using the local zeta function method it is easy to identify the divergences of the stress-energy tensor but the finite part is written in terms of expressions involving double summations. A simple trigonometric identity allow us to obtain expressions with only one summation. The advantage is that all the calculations can be done analytically. To regularize the one-loop vacuum fluctuations we use a mixed approach with analytic regularizations procedures. The form of surface divergences related with the uncertainly principle will be explicitly calculated. Some different physical arguments that support the introduction of surface countertems that remove these divergences will be discussed in the conclusions. It is clear that to obtain the renormalized vacuum expectation value of the stress-energy tensor is straightforward from our results. With the renormalized vacuum fluctuation of the field $\langle\varphi^2(x)\rangle_{ren}$, to find the renormalized stress-energy tensor we have only to apply certain second order differential operator depending on the type of the field under consideration in the renormalized two-point function, and after this the coincident point limit must be assumed.

The organization of the paper is the following: In the section II we study the one-loop vacuum fluctuations of a massive scalar field confined within a rectangular waveguide. In section III we use an analytic regularization method to identify the infinites that appear in the expression of the vacuum fluctuations of the massless field near the boundaries. Conclusions are given in section IV. In this paper we use $\hbar = c = 1$.

2 Vacuum fluctuations of a massless scalar field confined within a rectangular waveguide

In this section and in the next one we will investigate the the one-loop vacuum fluctuations $\langle\varphi^2(x)\rangle$ associated with a scalar field in the rectangular waveguide. We will suppose a four dimensional Euclidean space, where the last two coordinates are unbounded, while the first two, that we call

x_1 and x_2 lie in the interval $[0, a]$ and $[0, b]$ respectively. We assume Dirichlet boundary conditions on the boundaries. The free field is defined in the region

$$\Omega = \mathbf{x} \equiv (x_1, x_2, x_3, x_4) : 0 < x_1 < a, \quad 0 < x_2 < b \subset \mathbf{R}^4, \tag{2}$$

with Dirichlet boundary conditions at $x_1 = 0$ and $x_1 = a$ and also $x_2 = 0$ and $x_2 = b$. As we discussed, the lack of translational invariance introduce surfaces and edges divergences. One way to avoid these divergences is to smooth out the plates surface. In this case loop-graphs will depend on ad-hoc model assumption and consequently we prefer to mantain these hard walls. Nevertheless Ford and Svaiter [31] to solve a long standing paradox concerning the renormalized energy of minimally and conformally coupled scalar fields used a quantum mechanical treatment of the boundary conditions. Still in the context of the Casimir energy of minimally coupled scalar fields, many authors used soft, hard and semi-hard boundary conditions [49] [50] [51]. As we stressed before, we prefere, at least at the moment to keep only a hard classical boundary conditions, i.e. Dirichlet boundary conditions.

The basic idea of the Green's function method is the following: to find the renormalized vacuum expectation value of the stress-energy tensor we have to apply a certain second order differential operator in the renormalized two-point Green's function $G_{ren}(x, x')$. After this the coincident point limit is assumed. Note that the usual definition of the renormalized two point function is to subtract from the two-point Green's function in the presence of the boundary the free space two point function:

$$G_{ren}(x, x') = [G_{\partial\Omega}(x, x') - G_0(x, x')]. \tag{3}$$

Since, if we assume Dirichlet boundary conditions the two-point function $G_{\partial\Omega}(x, x')$ vanishes by construction on the boundaries, the renormalized two-point function develops the singularities of the free two-point function $G_0(x, x')$ at the coincident point $x \to x'$. Here we decide to use an alternative regularization procedure that also give us the surface and edge divergences related with the uncertainly relation between the field and the canonical conjugate momentum associated with the field. Consequently we start with the expresssion of the one-loop vacuum fluctuations of the massless scalar field, confined in the interior of a rectangular waveguide. Using analytic regularization procedures i.e. a mixed between dimensional and "zeta" function analytic regularization, the infinities related with the imposition of classical boundary conditions will be identified and can be eliminated by the introduction of surfaces and edges counterterms. The fundamental problem for us is the lack of translational invariance where Green's functions can only be expressed by infinite sums. In rectangular cavities a characteristic feacture of the Green's functions is that they are expressed in term of double summations in the discrete eigenfrequencies, but as we will see a simple trigonometric identity will allow us to obtain expressions that contain only one summation. It is very important for us to obtain manageable expressions.

Since we will use dimensional regularization, we will work at the begining in a d-dimensional Euclidean space. Although Minkowski fields do not admit analytic continuations, their vacuum expectation values do. It is well known that it is possible to analytic extend the n-point function or the Wightman functions defined in Minkowski spacetime by Euclidean counterpart, i.e. the n-point Schwinger functions. Our approach will be to calculate the two-point Schwinger function at coincident points (tadpole) in the interior of the waveguide submitted to Dirichlet boundary conditions in the walls, and then use analytic regularization to eliminate the ultraviolet divergences. In the one-loop vacuum fluctuation, $\langle\varphi^2(x_1,x_2,a,b)\rangle$ we will change the notation of the previous section to $x_1 = x$, $x_2 = y$. Let the waveguide be oriented along the z axis with walls at $x = 0$ and a and $y = 0$ and b. In this mixed representation, since we are assuming Dirichlet b.c., the expression for the one-loop vacuum fluctuations (with the external legs amputed), that we call $T_{DD}(x,y;a,b,d)$ is given by:

$$T_{DD}(x,y;a,b,d) = \frac{4}{(2\pi)^{d-2}ab}\sum_{n,n'=1}^{\infty}\sin^2(\frac{n\pi x}{a})\sin^2(\frac{n'\pi y}{b}) \int d^{d-2}p\frac{1}{\left(\vec{p}^2+(\frac{n\pi}{a})^2+(\frac{n'\pi}{b})^2+m^2\right)}. \tag{4}$$

There are two points that we would like to stress. First is the fact that to perform analytic regularizations we have to introduce a parameter μ with dimension of mass in order to have dimensionless quantities raised to a complex power. For sake of simplicity we omit the μ factor in the following. Second is the fact that the generalization for the case of Neumann b.c. is straightforward. In this case infrared divergences associated with the $n = 0$ mode will appear. Consequently, here, we introduced a mass parameter to control the infrared divergences. Thus all the expressions have a m dependence that we omit in the left hand side of each expression to simplify the notation. In the end of the calculation we will put $m = 0$. With these observations in mind, our purpose is to use an important idea of the multiple reflection expansion, used to perform perturbation theory in cavities [8] [45] [52]. A very important idea of the multiple reflection expansion for cavity propagators is the decomposition of the propagator into a free term and a boundary term. For arbitrary geometries this can not be obtained in a closed form, but for rectangular cavities it is possible to perform this approach. Using trigonometric identities it is straightforward to have :

$$T_{DD}(x,y;a,b,d) = T(a,b,d) + T(x;a,b,d) + T(y;a,b,d) + T(x,y;a,b,d), \tag{5}$$

where each expression of the above equation can be easily calculated. For $T(a,b,d)$ we have

$$T(a,b,d) = \frac{4}{(2\pi)^{d-2}ab} \sum_{n,n'=1}^{\infty} \int d^{d-2}p \frac{1}{\left(\vec{p}^2 + (\frac{n\pi}{a})^2 + (\frac{n'\pi}{b})^2 + m^2\right)}. \tag{6}$$

It is possible to show that $T(x;a,b,d)$ is given by

$$T(x;a,b,d) = -\frac{1}{2}T(a,b,d) + \frac{2}{(2\pi)^{d-2}ab} \sum_{n,n'=1}^{\infty} \int d^{d-2}p \frac{\cos(\frac{2n\pi x}{a})}{\left(\vec{p}^2 + (\frac{n\pi}{a})^2 + (\frac{n'\pi}{b})^2 + m^2\right)}. \tag{7}$$

The expression for $T(y;a,b,d)$ has the same functional form of the above equation only changing x by y . Consequently we have :

$$T(y;a,b,d) = -\frac{1}{2}T(a,b,d) + \frac{2}{(2\pi)^{d-2}ab} \sum_{n,n'=1}^{\infty} \int d^{d-2}p \frac{\cos(\frac{2n'\pi y}{b})}{\left(\vec{p}^2 + (\frac{n\pi}{a})^2 + (\frac{n'\pi}{b})^2 + m^2\right)}, \tag{8}$$

and finally :

$$T(x,y;a,b,d) = \frac{1}{(2\pi)^{d-2}ab} \sum_{n,n'=1}^{\infty} \int d^{d-2}p \frac{1}{\left(\vec{p}^2 + (\frac{n\pi}{a})^2 + (\frac{n'\pi}{b})^2 + m^2\right)} \left(1 - \cos(\frac{2n\pi x}{a}) - \cos(\frac{2n'\pi y}{b}) + \cos(\frac{2n\pi x}{a})\cos(\frac{2n'\pi y}{b})\right). \tag{9}$$

Collecting the expressions for $T(a,b,d)$, $T(x;a,b,d)$ and $T(y;a,b,d)$ it is easy to express $T(x,y;a,b,d)$ as

$$\begin{aligned} T(x,y;a,b,d) &= \frac{1}{4}T(a,b,d) - \frac{1}{2}\left[T(x,a,b,d) + \frac{1}{2}T(a,b,d)\right] + \\ & -\frac{1}{2}\left[T(y,a,b,d) + \frac{1}{2}T(a,b,d)\right] + N(x,y;a,b,d), \end{aligned} \tag{10}$$

where the expression for $N(x,y;a,b,d)$, i.e the part that contain edges divergences is given by:

$$N(x,y;a,b,d) = \frac{1}{(2\pi)^{d-2}ab}\sum_{n,n'=1}^{\infty}\int d^{d-2}p\frac{\left(\cos(\frac{2n\pi x}{a})\cos(\frac{2n'\pi y}{b})\right)}{\left(\vec{p}^2 + (\frac{n\pi}{a})^2 + (\frac{n'\pi}{b})^2 + m^2\right)}. \tag{11}$$

Let us study each contribution separately. Using dimensional regularization on eq.(6) it is possible to write :

$$T(a,b,d) = \frac{4}{(2\sqrt{\pi})^{d-2}}\Gamma(2-\frac{d}{2})\sum_{n,n'=1}^{\infty}\frac{1}{\left(m^2 + (\frac{n\pi}{a})^2 + (\frac{n'\pi}{b})^2\right)^{2-\frac{d}{2}}}. \tag{12}$$

The equation above is the part of the vacuum field fluctuation which does not depend from the distance to the boundaries and in the renormalization procedure will demand only bulk counterterm i.e. has a position independent divergent part. The structure of the divergences of the Epstein zeta function is well know in the literature. See for example Ford [53] and also Ford and Svaiter [54]. Nevertheless for the case that we are interested i.e. $T(a,b,d)$ in $d=4$ we have to use the analytic extension given by Ambjorn and Wolfram [42] or Kirsten [55].

Since the polar structure of $T(a,b,d)$ can be found in the literature to calculate the analytic structure of $T(x;a,b,d)$ we will concentrate only on the position dependent divergent part given by $T(x;a,b,d)+\frac{1}{2}T(a,b,d)$. This expression is given by

$$T(x;a,b,d)+\frac{1}{2}T(a,b,d)=\frac{2}{(2\pi)^{d-2}ab}\sum_{n,n'=1}^{\infty}\int d^{d-2}p\frac{\cos(\frac{2n\pi x}{a})}{\left(\vec{p}^2+(\frac{n\pi}{a})^2+(\frac{n'\pi}{b})^2+m^2\right)}. \tag{13}$$

Although eq.(13) is written in terms of two sums, one of the sums can be easily done using a trigonometric expression given by [56]

$$\sum_{n=1}^{\infty}\frac{\cos nt}{n^2+A^2}=-\frac{1}{2A^2}+\frac{\pi}{2A}\frac{\cosh A(\pi-t)}{\sinh \pi A}. \tag{14}$$

The usefulness of this trigonometric identity is related with the fact that it will remain only one summmation in all remaining expressions. The importance of this fact to the case of rectangular domains lies in the ability to analytically regularize the majority of the remaining divergent expressions. To show how does this works, let us proceed with the calculations.

After a long but straighforward calculation using eq.(14) it is possible to write eq.(13) as

$$T(x;a,b,d)+\frac{1}{2}T(a,b,d)=R_1(a,b,d)+R_2(x;a,b,d) \tag{15}$$

where :

$$R_1(a,b,d)=-\frac{1}{(2\pi)^{d-2}ab}\sum_{n'=1}^{\infty}\int d^{d-2}p\frac{1}{(\vec{p}^2+m^2+(\frac{n'\pi}{b})^2)} \tag{16}$$

and

$$R_2(x;a,b,d)=\frac{1}{(2\pi)^{d-2}b}\sum_{n'=1}^{\infty}\int d^{d-2}p\frac{1}{\sqrt{\vec{p}^2+m^2+(\frac{n'\pi}{b})^2}}\frac{\cosh((a-2x)\sqrt{\vec{p}^2+m^2+(\frac{n'\pi}{b})^2})}{\sinh(a\sqrt{\vec{p}^2+m^2+(\frac{n'\pi}{b})^2})}. \tag{17}$$

It is clear that to calculate the analytic structure for the case of the position dependent divergent part $T(y;a,b,d)$ we can use the same method that we use for $T(x;a,b,d)$. Consequently the expression for $T(y;a,b,d)+\frac{1}{2}T(a,b,d)$ is :

$$T(y;a,b,d)+\frac{1}{2}T(a,b,d)=I_1(a,b,d)+I_2(y;a,b,d) \tag{18}$$

where :

$$I_1(a,b,d)=-\frac{1}{(2\pi)^{d-2}\,ab}\sum_{n=1}^{\infty}\int d^{d-2}p\frac{1}{(\vec{p}^2+m^2+(\frac{n\pi}{a})^2)} \tag{19}$$

and

$$I_2(y;a,b,d)=\frac{1}{(2\pi)^{d-2}\,a}\sum_{n=1}^{\infty}\int d^{d-2}p\frac{1}{\sqrt{\vec{p}^2+m^2+(\frac{n\pi}{a})^2}}\frac{\cosh((b-2y)\sqrt{\vec{p}^2+m^2+(\frac{n\pi}{a})^2})}{\sinh(b\sqrt{\vec{p}^2+m^2+(\frac{n\pi}{a})^2})}. \tag{20}$$

Both $I_1(a,b,d)$ and $R_1(a,b,d)$ are the same and after a dimensional regularization procedure we will obtain a special Epstein-Hurwitz zeta function. The analytic extension of this function for general d in the massive and massless case can be found in the literature. For the massive case see for example [57]. For the massless case using dimensional regularization and the duplication formula for the gamma function it is possible to write

$$I_1(a,b,d)|_{m=0}=\frac{a^{3-d}}{b}f_1(d)\zeta(4-d)\Gamma(4-d), \tag{21}$$

where

$$f_1(d)=-\frac{1}{2}\frac{\pi^{\frac{d-5}{2}}}{\Gamma(\frac{5-d}{2})}, \tag{22}$$

is an entire function of d. Since the $\zeta(z)$ and $\Gamma(z)$ can be analytically continued from an open connected set of points in the complex z plane into the entire domain of z, it is easy to find the analytic structure of $I_1(a,b,d)|_{m=0}$. To find the analytic structure of $I_2(y;a,b,d)$ and $R_2(x;a,b,d)$

let us concentrate on $I_2(y;a,b,d)$. Integrating over the solid angle in eq.(20), i.e using the fact that $d^dp = p^{d-1}dp\, d\Omega_d$ and $\int\, d\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$ we have :

$$I_2(y;a,b,d) = \frac{1}{a}h(d)\sum_{n=1}^{\infty}\int dp\, p^{d-3}\frac{1}{\sqrt{\vec{p}^2+m^2+(\frac{n\pi}{a})^2}}\frac{\cosh((b-2y)\sqrt{\vec{p}^2+m^2+(\frac{n\pi}{a})^2})}{\sinh(b\sqrt{\vec{p}^2+m^2+(\frac{n\pi}{a})^2})} \tag{23}$$

where :

$$h(d) = \frac{2}{(2\sqrt{\pi})^{\frac{d-2}{2}}\Gamma(\frac{d-2}{2})}. \tag{24}$$

Performing a change of variables $v = \left(\vec{p}^2+m^2+(\frac{n\pi}{a})^2\right)^{\frac{1}{2}}$, and now going back to the physical case $d=4$, is possible to write $I_2(y,a,b,d)|_{d=4} \equiv I_2(y;a,b)$ as:

$$I_2(y;a,b) = \frac{1}{a}h(4)\sum_{n=1}^{\infty}\int_{\alpha}^{\infty} dv\, \frac{\cosh((b-2y)v)}{\sinh bv} \tag{25}$$

where the lower limit of the above integral is given by :

$$\alpha = \sqrt{m^2+(\frac{n\pi}{a})^2}. \tag{26}$$

Note that the situation is completely different for $d \neq 4$, since it will appear in the integrand of eq.(23) $(v^2-m^2-(\frac{n\pi}{a})^2)^{\frac{d-4}{2}}$. As a consequence of this fact, it is more dificult to perform algebraic manipulations that allow us to analytically regularize $R_2(x,a,b,d)$ and $I_2(y,a,b,d)$ and also other expressions. Nevertheless, it is possible to extend our results to $d>4$. The expression given by eq.(23), after change of variables contains a power of a binomial. When d is even (for $d>4$) the power is an integer and the use of the Newton's binomial theorem will give a very direct way to generalize our further results. When d is odd ($d>4$), the expansion on the binomial yields an infinite power series. Using the same techniques used by Svaiter and Svaiter [58] it is possible to regularize eq.(23) and other expressions for the general case $d \neq 4$. The generalization for arbitrary d is under investigation by the authors. These arguments will be clarified in the next section. Using trigonometric identities, we have :

$$I_2(y;a,b) = I_{21}(y;a,b) + I_{22}^{+}(y;a,b) + I_{22}^{-}(y;a,b) \tag{27}$$

where :

$$I_{21}(y;a,b) = \frac{1}{2ya}h(4)\sum_{n=1}^{\infty} exp\left(-2y\sqrt{m^2+(\frac{n\pi}{a})^2},\right) \tag{28}$$

$$I_{22}^{+}(y;a,b) = \frac{1}{2a}h(4)\sum_{n=1}^{\infty}\int_{b\alpha}^{\infty} dq\,(\coth q - 1)\, e^{2\frac{y}{b}q} \tag{29}$$

and finally

$$I_{22}^{-}(y;a,b) = \frac{1}{2a}h(4)\sum_{n=1}^{\infty}\int_{b\alpha}^{\infty} dq\,(\coth q - 1)\, e^{-2\frac{y}{b}q}. \tag{30}$$

An exact expression in the massless case can be obtained from $I_{21}(y;a,b)$. Summing the geometric series it is possible to find $I_{21}(y;a,b)|_{m=0}$, i.e

$$I_{21}(y;a,b)|_{m=0} = \frac{1}{2ya}h(4)\frac{1}{e^{\frac{2\pi}{a}y}-1}. \tag{31}$$

The expression $I_{22}^{-}(y;a,b)$ is convergent and $I_{22}^{+}(y;a,b)$ has a pole at $y=b$. The polar structure of $I_2(y;a,b)$ and also $R_2(x;a,b)$ will be discussed in the next section. We still have to calculate $N(x,y;a,b)$. Using again the trigonometric identity given by eq.(14) it is possible to write $N(x,y;a,b,d) = N_1(x;a,b,d) + N_2(x,y;a,b,d)$ where we have:

$$N_1(x;a,b,d) = -\frac{1}{2(2\pi)^{d-2}ab}\sum_{n=1}^{\infty}\int d^{d-2}p\frac{\cos(\frac{2n\pi x}{a})}{\left(\vec{p}^2+m^2+(\frac{n\pi}{a})^2\right)} \tag{32}$$

and

$$N_2(x,y;a,b,d) = \frac{1}{(2\pi)^{d-2}ab}\frac{b}{2\pi}\sum_{n=1}^{\infty}\int d^{d-2}p\frac{\cos(\frac{2n\pi x}{a})}{\sqrt{\left(\vec{p}^2+m^2+(\frac{n\pi}{a})^2\right)}}$$
$$\frac{\cosh((b-2y)\sqrt{\vec{p}^2+m^2+(\frac{n\pi}{a})^2})}{\sinh(b\sqrt{\vec{p}^2+m^2+(\frac{n\pi}{a})^2})}. \quad (33)$$

Let us study the expression given by $N_1(x;a,b,d)$. Using again eq.(14) it is possible to write $N_1(x;a,b,d) = N_{11}(a,b,d)+N_{12}(x;a,b,d)$, where $N_{11}(a,b,d)$ and $N_{12}(a,b,d)$ are given respectively by

$$N_{11}(a,b,d) = \frac{1}{4(2\pi)^{d-2}ab}\int d^{d-2}p\frac{1}{(\vec{p}^2+m^2)}. \quad (34)$$

and

$$N_{12}(x;a,b,d) = -\frac{1}{4(2\pi)^{d-2}b}\int d^{d-2}p\frac{1}{\sqrt{\vec{p}^2+m^2}}\frac{\cosh((a-2x)\sqrt{(\vec{p}^2+m^2)})}{\sinh(a\sqrt{\vec{p}^2+m^2})}. \quad (35)$$

The expression given by eq.(34) can be easily calculated using dimensional regularization. A straightforward calculation gives:

$$N_{11}(a,b,d) = \frac{1}{4ab(2\sqrt{\pi})^{d-2}}\Gamma(2-\frac{d}{2})(m^2)^{\frac{d}{2}-2}. \quad (36)$$

We have to deal with the expression of $N_{12}(x,y;a,b,d)$. Integrating over the solid angle, changing the variables and using the fact that we are in $d=4$, i.e defining $N_{12}(x;a,b,d)|_{d=4} \equiv N_{12}(x;a,b)$ we have

$$N_{12}(x;a,b) = -\frac{1}{8\pi b}\int_m^{\infty} dv\frac{\cosh((a-2x)v)}{\sinh av} \quad (37)$$

Again, using trigonometric identities, we have :

$$N_{12}(x;a,b) = -\frac{1}{8\pi ab}\left[\frac{a}{2x}e^{-2xm}+\frac{1}{2}\int_{am}^{\infty}dq\,(\coth q-1)\,e^{2q\frac{x}{a}}+\frac{1}{2}\int_{am}^{\infty}dq\,(\coth q-1)\,e^{-2q\frac{x}{a}}\right] \quad (38)$$

We note that eq.(38) has surface divergences at $x = 0$ and $x = a$. The structure of the divergences of $N_{12}(x; a, b)$ will be analyzed further.

As a next step in the discussion, let us investigate the part of the one-loop vacuum fluctuations that contains edge divergences, i.e $N_2(x, y; a, b)$. Consequently we will still have to study the polar structure of $N_2(x, y; a, b)$. Again, integrating over the solid angle, changing the variables and using the fact that we are in $d = 4$, the expression for $N_2(x, y; a, b)|_{d=4} \equiv N_2(x, y; a, b)$ is given by

$$N_2(x, y; a, b) = \frac{1}{4a} \sum_{n=1}^{\infty} \cos(\frac{2n\pi x}{a}) \int_{\alpha}^{\infty} dv \, \frac{\cosh((b - 2y)v)}{\sinh bv} \tag{39}$$

where the lower limit of the above integral is given by $\alpha = \sqrt{m^2 + (\frac{n\pi}{a})^2}$.

Using trigonometric identities it is possible to write $N_2(x, y; a, b)$ as

$$N_2(x, y; a, b) = N_{21}(x, y; a, b) + N_{22}(x, y; a, b), \tag{40}$$

where :

$$N_{21}(x, y; a, b) = -\frac{1}{8ay} \sum_{n=1}^{\infty} \cos(\frac{2n\pi x}{a}) e^{-2y\alpha} \tag{41}$$

and

$$N_{22}(x, y; a, b) = \frac{1}{4a} \sum_{n=1}^{\infty} \cos(\frac{2n\pi x}{a}) \int_{\alpha}^{\infty} dv \, (\coth(bv) - 1) \cosh 2vy. \tag{42}$$

From now we will put $m = 0$. We note that eq.(41) has two edges divergences. One at $x = y = 0$ and the other at $x = a$, $y = 0$. To see the divergences of eq.(42) we rewrite $N_{22}(x, y; a, b)$ as

$$N_{22}(x, y; a, b) = N_{22}^{+}(x, y; a, b) + N_{22}^{-}(x, y; a, b) \tag{43}$$

where

$$N_{22}^{+}(x,y;a,b) \;=\; \frac{1}{8ab}\sum_{n=1}^{\infty}\cos(\frac{2n\pi x}{a})\int_{\alpha b}^{\infty}dq\;(\coth q-1)\,e^{2q\frac{y}{b}} \tag{44}$$

$$N_{22}^{-}(x,y;a,b) \;=\; \frac{1}{8ab}\sum_{n=1}^{\infty}\cos(\frac{2n\pi x}{a})\int_{\alpha b}^{\infty}dq\;(\coth q-1)\,e^{-2q\frac{y}{b}}. \tag{45}$$

We note that $N_{22}^{+}(x,y;a,b)$ has two edge divergences, one at $x=0$, $y=b$ and the other at $x=a$, $y=b$. We note also that $N_{22}^{-}(x,y;a,b)$ is finite everywhere. The analytic structure of the vacuum field fluctuations $\langle\varphi^2(x)\rangle$ will be analysed in the next section.

3 Analysis of ultraviolet divergences of $T_{DD}(x,y;a,b,d)$

The purpose of this section is to analyze the general structure of the ultraviolet divergences of $\langle\varphi^2(x)\rangle$. For the construction of the renormalized theory it is not suffices to known that such counterterms do exist, we have to known explicity the expressions of the bulk and surface counterterms. As we discussed in the previous section it is possible to write the one-loop vacuum fluctuations $T_{DD}(x,y;a,b,d)$ as

$$T_{DD}(x,y;a,b,d)=T(a,b,d)+T(x;a,b,d)+T(y;a,b,d)+T(x,y;a,b,d). \tag{46}$$

The first expression that we have to deal is $T(a,b,d)$. As we discussed in the previous section the analytic structure of $T(a,b,d)$ was carefully analysed by Kirsten [55] and it is not necessary to repeat the calculation again. The second term of the above equation and third one can be written respectively in terms of $R_1(a,b,d)$, $R_2(x;a,b,d)$, $I_1(a,b,d)$ and finally $I_2(y;a,b,d)$. We would like to stress that again the polar structure of $R_1(a,b,d)$ and $I_1(a,b,d)$ can be found in the refs. [53] and [55] and we will not repeat the analysis that was done in these papers. The next quantity that we have to regularize is $R_2(x;a,b,d)$ and also $I_2(y;a,b,d)$. Since both cases are similar let us study only the expression given by $R_2(x;a,b,d)$. We are interested in studying the form of the divergences near the plates ($x \to 0$ *and* $x \to a$), consequently let us assume $b \gg a$. In this case it is possible to write $R_2(x;a,b,d)|_{b\gg a}$. Let us define $R_2(x;a,b,d)|_{b\gg a} = r_2(x;a,d)$, thus:

$$r_2(x;a,d) = \frac{1}{2\,(2\pi)^{d-1}} \int d^{d-1}p \frac{1}{\sqrt{\vec{p}^2+m^2}} \frac{\cosh((a-2x)\sqrt{\vec{p}^2+m^2})}{\sinh(a\sqrt{\vec{p}^2+m^2})}. \tag{47}$$

We will again use the fact that $d^{d-1}p = p^{d-2}dp\,d\Omega_{d-1}$ and $\int d\Omega_{d-1} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}$. Let us choose now $m=0$, and for reasons that will become evident latter, we must have $d>3$. Defining $h_2(d)$ by:

$$h_2(d) = \frac{1}{2^{d-2}\pi^{\frac{d-1}{2}}\Gamma(\frac{d-1}{2})}, \tag{48}$$

it is possible to write $r_2(x;a,d)|_{m=0}$ as

$$\begin{aligned} r_2(x;a,d)|_{m=0} &= \frac{1}{2}h_2(d)\int_0^\infty dk\,k^{d-3}\coth ka\cosh 2kx \\ &- h_2(d)\int_0^\infty dk\,k^{d-3}\cosh kx\sinh kx. \end{aligned} \tag{49}$$

Let us assume $x \neq 0$ and $x \neq a$. A straightforward calculation gives

$$\begin{aligned} r_2(y;a,d)|_{m=0} &= \frac{1}{2}h_2(d)\Big[\int_0^\infty dk\,k^{d-3}(\coth ka-1)\cosh 2ky \\ &+ \int_0^\infty dk\,k^{d-3}(\cosh 2ky - \sinh 2ky)\Big]. \end{aligned} \tag{50}$$

In the first integral for large k, $(\coth ka - 1)$ has the behavior: $(\coth ka - 1) \sim e^{-2ka}$. Moreover, the second integral in the above equation is ultraviolet finite for $x \neq 0$. Let us define $t = ka$ and $q = kx$ in the first and second integrals above respectively. Then Eq.(50) becomes:

$$
\begin{aligned}
r_2(x; a, d)|_{m=0} &= \frac{1}{2a^{d-2}} h_2(d) \int_0^\infty dt\, t^{d-3} (\coth t - 1) \cosh(\frac{2x}{a} t) \\
&+ \frac{1}{2x^{d-2}} h_2(d) \int_0^\infty dq\, q^{d-3} (\cosh 2q - \sinh 2q).
\end{aligned}
\tag{51}
$$

The second term in the above equation gives us the well known result that for a massless scalar field $\langle \varphi^2(x) \rangle$ diverges as $\frac{1}{x^2}$ (in a four-dimensional space) as we approach the plate [21]. In order to analyze the polar part of $r_2(x; a, d)$, we use the definition of the Gamma function. Let us define $J_1(\nu, \mu)$ and $J_2(\mu, \alpha)$ by

$$
J_1(\mu, \nu) = \int_0^\infty dt\, t^{\mu-1} e^{-\nu t} = \frac{1}{\nu^\mu} \Gamma(\mu), \quad Re(\mu) > 0, \;\; Re(\nu) > 0 \tag{52}
$$

and

$$
J_2(\mu, \alpha) = \int_0^\infty dt\, t^{\mu-1} e^{-\alpha t} (\coth t - 1) = 2^{1-\mu} \Gamma(\mu) \zeta(\mu, \frac{\alpha}{2} + 1) \;\; Re(\alpha) > 0, \;\; Re(\mu) > 1, \tag{53}
$$

where $\zeta(s, u)$ is the Hurwitz zeta function defined by [56]

$$
\zeta(s, u) = \sum_{n=0}^\infty \frac{1}{(n+u)^s}, \quad Re(s) > 1, \quad u \neq 0, -1, -2... \tag{54}
$$

Then, using eqs.(52), (53) and (54) in eq.(51) we have that:

$$
\begin{aligned}
r_2(x; a, d)|_{m=0} &= \frac{1}{2} h_2(d) \frac{1}{a^{d-2}} \left[2^{2-d} \Gamma(d-2) \left(\zeta(d-2, \frac{x}{a} + 1) + \zeta(d-2, -\frac{x}{a} + 1) \right) \right] \\
&+ \frac{1}{(2x)^{d-2}} h_2(d) \Gamma(d-2).
\end{aligned}
\tag{55}
$$

Using the definition of the zeta function, it is evident that:

$$\begin{aligned}
&\frac{1}{a^{d-2}}\left(\zeta(d-2,\frac{x}{a}+1)+\zeta(d-2,-\frac{x}{a}+1)\right)= \\
&\frac{1}{a^{d-2}}\sum_{n=0}^{\infty}\frac{1}{\left(n+(1+\frac{x}{a})\right)^{d-2}}+\frac{1}{(a-x)^{d-2}}+\frac{1}{a^{d-2}}\sum_{n=1}^{\infty}\frac{1}{\left(n+(1-\frac{x}{a})\right)^{d-2}}.
\end{aligned} \tag{56}$$

We see that the regularized $r_2(x;a,d)|_{m=0}$ has two poles of order $(d-2)$ in $x=0$ and in $x=a$. Note that the residues of the poles in $x=0$ and in $x=a$ are a-independent. The same analysis can be done for $I_2(y;a,b,d)$ assuming $a \gg b$.

Let us finally analyze $N_{21}(x,y;a)$ and $N_{22}(x,y;a,b)$, given respectively by

$$N_{21}(x,y;a) = -\frac{1}{8ay}\sum_{n=1}^{\infty}\cos(\frac{2n\pi x}{a})e^{-\frac{2n\pi y}{a}} \tag{57}$$

and

$$N_{22}(x,y;a,b) = \frac{1}{4ab}\sum_{n=1}^{\infty}\cos(\frac{2n\pi x}{a})\int_{\frac{n\pi}{a}b}^{\infty}dq\,(\coth q-1)\cosh(2q\frac{y}{b}). \tag{58}$$

To find the analytic structure of $N_{21}(x,y;a)$ we can expand the general term in the sum in power series, commute the two summations, and use analytic continuation in the zeta function that will appear. The process will in general produce an extra term, which is generated by commuting the convergent exponential summation $\sum_m$ with the new divergent summation $\sum_k$ (for details see for e.g. refs. [59][60][61]). In our case this term vanishes due to the power of n. Let us express the sum that appears in eq.(3) in terms of the complex variable $z=ix-y$:

$$N_{21}(x,y;a) = -\frac{1}{8ay}Re\left\{\sum_{n=1}^{\infty}\exp\left(\frac{2n\pi z}{a}\right)\right\}. \tag{59}$$

Expanding around $z = 0$ will produce :

$$N_{21}(x, y; a) = -\frac{1}{8ay} Re \left\{ \sum_{k=0}^{\infty} \left(\frac{2\pi}{a}\right)^k \frac{(z)^k}{k!} \zeta(-k) + \left(\frac{a}{2\pi}\right) z^{-1} - 1 \right\}. \tag{60}$$

We see that the edge divergence appears in the term $1/z$. Taking the real part :

$$N_{21}(x, y; a) = \frac{1}{16\pi\,(y^2 + x^2)} - \frac{1}{8ay} Re\,\{f_1(z)\}\,, \tag{61}$$

where $f_1(z)$ is an entire function of z and is given by :

$$f_1(z) = Re \left\{ \sum_{k=0}^{\infty} \left(\frac{2\pi}{a}\right)^k \frac{(z)^k}{k!} \zeta(-k) \right\}. \tag{62}$$

Expanding around $z = ia$ will produce :

$$N_{21}(x, y; a) = -\frac{1}{8ay} Re \left\{ \sum_{k=0}^{\infty} \left(\frac{2\pi}{a}\right)^k \frac{(z - ia)^k}{k!} \zeta(-k) + \left(\frac{a}{2\pi}\right) (z - ia)^{-1} - 1 \right\}. \tag{63}$$

Taking the real part of $1/(z - ia)$, we have :

$$N_{21}(x, y; a) = \frac{1}{16\pi\,(y^2 + (x - a)^2)} - \frac{1}{8ay} f_2(z), \tag{64}$$

where $f_2(z)$ is also an entire function of z and is given by:

$$f_2(z) = Re \left\{ \sum_{k=0}^{\infty} \left(\frac{2\pi}{a}\right)^k \frac{(z - ia)^k}{k!} \zeta(-k) \right\}. \tag{65}$$

To find the analytic structure of $N_{22}(x, y; a, b)$, it is enough to analyse the quantity $N_{22}^{+}(x, y; a, b)$, which is given by the eq.(44) (the other quantity $N_{22}^{-}(x, y; a, b)$ is finite everywhere). To calculate the integral in eq.(44), we can express $\coth bv$ with exponencial functions and expand the integrand in power series. The integral can be easily evaluated and the result is given by

$$N_{22}^{+}(x,y;a,b) = -\frac{1}{8ab}\sum_{n=1}^{\infty}\cos(\frac{2n\pi x}{a})\sum_{k=0}^{\infty}\frac{e^{-2\left(1-\frac{y}{b}+k\right)\frac{n\pi}{a}b}}{\left(1-\frac{y}{b}+k\right)}. \tag{66}$$

This result can be written in terms of the LerchPhi function which is defined as follows [62]:

$$LerchPhi\,(z,c,v) = \sum_{k=0}^{\infty}\frac{z^k}{(v+k)^c}. \tag{67}$$

which is valid for $abs(z) < 1$. By analytic continuation it is extended to the whole complex plane. This function has singularities at $z = 1$ and $c = 0$ or $c = 1$ and when v is a non-positive integer and $\mathrm{Re}(c)$ is also non-positive. We note that only the term $k = 0$ has a surface divergence at $y = b$. The remaing part of the series in k is finite everywhere and we call it $F(x,y;a,b)$. Using the same procedure used to analyse $N_{21}(x,y;a)$, we can define $z = (b-y) + ix$ and expand the term $k = 0$ around $z = 0$ or around $z = ia$. Therefore we have

$$N_{22}^{+}(x,y;a,b) = B(x,y;a,b) + F(x,y;a,b), \tag{68}$$

where the first expansion $B(x,y;a,b)$ is given by

$$B(x,y;a,b) = \frac{1}{16\pi\left(x^2+(y-b)^2\right)} + \frac{1}{8ab}h_1(z) \tag{69}$$

where

$$h_1(z) = Re\left\{\sum_{k=0}^{\infty}\left(\frac{2\pi}{a}\right)^k\frac{(z)^k}{k!}\zeta(-k)\right\}. \tag{70}$$

We note that in this case $B(x,y;a,b)$ has a edge divergence for $x = 0$, $y = b$. For the second expansion, $B(x,y;a,b)$ is given by

$$B(x,y;a,b) = \frac{1}{16\pi\left((x-a)^2+(y-b)^2\right)} + \frac{1}{8ab}h_2(z) \tag{71}$$

where

$$h_2(z) = Re\left\{\sum_{k=0}^{\infty}\left(\frac{2\pi}{a}\right)^k \frac{(z-ia)^k}{k!}\zeta(-k)\right\}. \tag{72}$$

We note that in this case $B(x, y; a, b)$ has a edge divervence for $x = a$, $y = b$. The behaviour of all surface divergences was predicted many years ago [5]. A general picture that emerges from the above discussion is the following: we have found that in order to eliminate the ultraviolet divergences of the one-loop vacuum fluctuations we have to introduce not only the usual counterterms but also counterterms concentrated on the boundaries. Since we have the renormalized vacuum fluctuations, it is possible to find the renormalized vacuum expectation value of the stress-energy tensor as we discussed before.

4 Discussions and conclusions

In this paper we obtained a renormalized closed form of the one-loop vacuum fluctuations associated with a massless scalar field defined in the interior of a infinity long waveguide. We would like to stress that our first basic assumption was that the scalar field is confined in the interior of the waveguide. This is like the MIT bag model assumption. For a recent study of massless fermions confined between paralel plates in a D-dimensional spacetime, see for example ref. [63]. In the case of the local problem, surface and edge divergences will appear related with the uncertaintly principle. It is natural to ask how it is possible to throw way these surface divergences. There are at least three different possible solutions that can eliminate these divergences:

i) The inclusion of the external modes assuming hard boundary conditions,

ii) taking into account the real properties of material i.e imperfect conductivity at high frequencies, or

iii) a quantum mechanical treatment of the boundary conditions, as was done by Ford and Svaiter [31].

Before discuss some local experiments that can measure the vacuum fluctuations, we would like to make some comment concerning (*i*), (*ii*) and (*iii*). Concerning the first possibility, i.e (*i*), one point that we wish to mention is that there is no confinement for the scalar field in the world. In other words, the physical situation is the existence of interior and exterior modes. We assumed the bag model configuration, since the exterior modes problem is still unsolved. Nevertheless, we believe that there must have a cancelation of the surface and edges infinities, if the exterior modes are taken into account. Some authors showed that surface divergences that appear in the energy density reverses sign in the other side of the boundary. This fact makes we thought that there must have a cancelation of the surface and edges infinities, if the exterior modes are taken into account. We would like to point out that in the bag situation there also appear infinities related with the volume, area, etc of the domain where the fields are confined. It is clear that in this case the divergences that appear are related with the Weyl theorem, i.e. the asymptotic distribution of the eigenvalues of some eliptic diferential operator.

Concerning the second possibility, i.e (*ii*), as was stressed by many authors the infinities that appear in renormalized values of local observables for the ideal conductor (or perfect mirror) represent a breakdown of the perfect-conductor approximation. A wavelength cutoff corresponding to the finite plasma frequency must be included. In this approach some questions arise. Does this approach produce terms in the vacuum energy that depend on the molecular properties of the boundaries? Do these terms depend on the geometry or not? In the case of the sphere Candelas [64] founded a cutoff dependent geometry-independent term. Consequently this term can not give any contribution to the global vacuum stress.

Finally an alternative choice is given by (iii). From ref.[31] we learn that position fluctuations of a reflecting boundary also remove divergences in the renormalized values of local observables. Here we use an analytic regularization procedure to identify these divergent terms. As we discussed, surface and edge counterterms (i.e. counterterms concentrated on the boundaries) produce a finite $\langle\varphi^2(x)\rangle$ on the boundaries. Both (ii) and (iii) give plausible physical arguments concerning the finiteness of local observables.

Although the global Casimir effect is more related to experiments where we measure the force between macroscopic surfaces, the local properties of the vacuum field fluctuations can in principle be observed by measuring the energy level shift of an atom interacting with an electromagnetic field. The distortion of the vacuum fluctuations due to the presence of classical boundaries can also be measured studying the spontaneous and induced emission of excited atoms in the presence of classical boundaries. In others words an atom travelling in the interior of a long waveguide can be used to test the correctness of our calculations. Some papers that go in this direction are refs.[65] [66] [67]. More recently the probability for unit time of spontaneous emission of a two level system coupled to a massless scalar field via a monopole interaction hamiltonian was calculated by Ford et al [68]. These authors studied spontaneous and induced emission of the two level system in the presence of one or two infinite perfectly reflecting plates at zero and finite temperature.

At this point it is worth commenting upon the relationship between the scalar model and the more realistic case of an atom wich interacts with an electromagnetic field via an electric dipole coupling $-\vec{d}.\vec{E}$, where $\vec{d}$ is the eletric dipole moment operator and $\vec{E}$ is the eletric field operator at the position of the point atom. After the use of first order perturbation theory it is possible to show that the probability that an electron makes a transition from one state to another is related to the Fourier transform over the observation time of normal ordered field correlation function.

Consequently, the generalization of the calculation that we did for the case of the eletromagnetic field is desirable. In this case it is important to study cavity QED effects, i.e.. how the radiative corrections are affected by the cavity. A complete review of cavity QED can be found in ref. [69]. It is clear that from the arguments that we developed above, the rate of spontaneous emission of atoms traveling in a long waveguide can be controled. In the case of the waveguide it is interesting to study the rate of spontaneus emission changing the ratio $\rho = \frac{a}{b}$. For $a \gg b$ we have the situation of atoms between paralel mirrors.

Another direction still in the one-loop approximation is to study the interacting $\lambda\varphi^4$ theory. The renormalization of the four-point function in systems with breaking of translational invariance deserves further investigations. Also it is interesting to analyse the renormalization program if we take into account high order loops, where overlaping divergences emerge. These two subjects are under the investigation by the authors.

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